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AN IMPOSSIBILITY THEOREM FOR SPATIAL MODELS

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SOCIAL SCIENCE WORKING PAPER 437

August 1982

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INTRODUCTION

Arrow's impossibility theorem (Arrow [1]) concerns the inability to find a "satisfactory" procedure of aggregating individual preferences over social alternatives into a social ordering. There are five basic requirements for satisfactoriness imposed by the theorem. The first is what has become known as collective rationality, that is, the requirement that the social ordering so derived be transitive. The second is that of unrestricted domain, namely that the procedure ought to work for a society whose members may have any transitive preference ordering. The third requirement is known as the Pareto principle, and states that if the members of society are unanimous in preferring one alternative to another, then the social ordering must also prefer it. The fourth condition, known as the requirement of independence of irrelevant alternatives, allows the social ordering of any pair of alternatives to reflect only the individual orderings over the pair, without reference to any other alternatives. The fifth condition is that of nondictatorship, that is, the procedure may not select an individual in advance and use his or her preferences for every realization of individuals' preferences. The impossibility theorem asserts that if there are at last three alternatives, then it is impossible to satisfy all five requirements simultaneously.

In an effort to define "satisfactoriness" in a way that admits the existence of satisfactory procedures, various authors have weakened or eliminated some of the above conditions. The contributions in these areas are too numerous to catalog. The approach taken in this paper is to weaken the requirement of unrestricted domain. Social scientists frequently make assumptions about the nature of individual preferences in formulating their models. For instance, economists view the set of alternatives as a set of allocations and assume that individuals have convex, monotonic, selfish preferences. Restricted domains of this sort have been studied by Kalai, Muller and Satterthwaite [7], Maskin [10] and Border [3]. The conclusions of these papers indicate that the sorts of restrictions on preferences used by economists do not vitiate the result of the impossibility theorem.

Another class of preferences are those used by political scientists in modeling electoral competition. In these models, the set of alternatives is an m-dimensional space of policy positions and voters are assumed to have type one preferences (Riker and Ordeshook [12]), i.e., preferences with a favorite point such that points further away are less preferred. (This is a real restriction in that not every triple is free. Given three colinear points, no type one preference ranks the middle point last.) In the case of a one-dimensional policy space, the preferences are "single-peaked" and it is well-known (Arrow [1], Black [2]) that simple majority rule provides a satisfactory procedure. In a space of more than one dimension, Kramer [9] pointed out that the single-peakedness condition is quite restrictive and is not the same as a restriction to type one preferences. McKelvey [11] demonstrated the rather dramatic failure of simple majority rule to be transitive in more than one dimension. The failure of simple majority rule to be satisfactory does not however rule out the existence of other satisfactory procedures. We prove below that even with a restriction to type one preferences there are no satisfactory procedures for aggregating individual orderings.

Another important paper dealing with restricted domains is Kalai and Muller [6]. They characterize all the domains which admit satisfactory procedures, provided that individuals or society are never allowed to be indifferent. This condition makes it impossible to apply their result to type one preferences which have spherical indifference surfaces. It is also not apparent how to carry their proof over to the case where there may be indifference. The proof presented below makes use of their notion of decisiveness implications, but is more closely related in structure to the proofs of Hansson [5], Kirman and Sondermann [8] and Border [3].

NOTATION AND DEFINITIONS

The set of alternatives is an m-dimensional euclidean space $\mathbb{R}^m$ with norm $|x| = [\sum_{i=1}^{m}(x_i^2)]^{1/2}$. A preference is a total transitive ordering on $\mathbb{R}^m$, the set of all preferences is denoted $\underline{P}$. Given a preference $R \in \underline{P}$, the strict preference associated with R is denoted

P. A preference R is of type one if there is some point p such that $x \, R \, y$ if and only if $|p - x| \leq |p - y|$. Such a point p is unique and is called the ideal point of R. The preference with ideal point p will be denoted R^p, and its strict preference will be denoted P^p. The class of all type one preferences will be denoted by D.

The set of voters in society will be denoted by N. An N-tuple $\langle R_i \rangle \in D^N$ is called a profile of individual preferences. A social welfare function for type one preferences is a function $f : D^N \to \underline{P}$. This definition embodies the domain restriction and collective rationality requirement. Note that society itself is not required to have type one preferences. The preference $f(\langle R_i \rangle)$ is the social ordering of alternatives for the profile where each voter $i \in N$ has preference R_i. When it is not likely to lead to confusion, we will refer to the social ordering $f(\langle R_i \rangle)$ as R and its strict preference simply by P; $f(\langle R_i' \rangle)$ will be denoted R', etc.)

An Arrow-type social welfare function is a social welfare function satisfying the following two conditions:

The Pareto Principle. If $x \, P_i \, y$ for all $i \in N$, then $x \, P \, y$.

Independence of Irrelevant Alternatives. If for all $i \in N$, $x \, R_i \, y \iff x \, R_i' \, y$, then $x \, R \, y \iff x \, R' \, y$.

A social welfare function is dictatorial if there is some $j \in N$ such that for every profile $\langle R_i \rangle$ and every pair of alternatives (x,y), if $x \, P_j \, y$, then $x \, P \, y$. Voter j is called the dictator.

THEOREM AND PROOFS.

Theorem. If the dimension m of the space of alternatives satisfies $m \geq 2$ and the set N of voters is finite, then every Arrow-type social welfare function for type one preferences is dictatorial.

Proof of the theorem. The proof of the theorem is broken down into several lemmas. In order to facilitate the proof we first introduce some more definitions.

A coalition is a subset of N. A coalition V is coercive for x over y if $x \neq y$ and whenever $x P_i y$ for all $i \in V$ and $y P_i x$ for all $i \in V^c$, then $x P y$. Coalition V is decisive for x over y if $x \neq y$ and whenever $x P_i y$ for all $i \in V$, then $x P y$. A coalition is decisive (resp. coercive) if it is decisive (resp. coercive) for x over y for every ordered pair (x,y) of distinct alternatives. Observe that if there is a voter who is a decisive coalition by himself, then he is a dictator.

The strategy of proof will be to characterize the class of decisive coalitions and show that one decisive coalition consists of a lone voter.

The following notion is borrowed from Kalai and Muller [6]. Let S be a collection of ordered pairs of distinct alternatives. We say that S is closed under coerciveness implication if the following condition C. holds.

Condition C. If there are type one preferences $R_1, R_2 \in D$ with $x P_1 y P_1 z$ and $y P_2 z P_2 x$, then

$$a). \quad (x,y) \in S \Rightarrow (x,z) \in S$$

and

$$b). \quad (z,x) \in S \Rightarrow (y,x) \in S.$$

Lemma 1. Let V be a coalition and let $S = \{(x,y) : V$ is coercive for x over y$\}$. Then S is closed under coerciveness implications.

Proof: Let $R_1, R_2 \in D$ satisfy $x P_1 y P_1 z$ and $y P_2 z P_2 x$.

a). Suppose $(x,y) \in S$, i.e., V is coercive for x over y. Define the profile $\langle R_i \rangle$ by

$$R_i = \begin{cases} R_1 & i \in V \\ R_2 & i \in V^c. \end{cases}$$

We can represent the preferences for members of coalitions V and V^c schematically as follows.

V	V^c
x	y
y	z
z	x.

Since V is coercive for x over y we have $x P y$. It follows from the Pareto principle that $y P z$ (as $y P_i z$ for all i). Thus by transitivity of the social ordering $x P z$. But $x P_i z$ for all $i \in V$ and $z P_i x$ for all $i \in V^c$, so by definition V is coercive for x over z, i.e., $(x,z) \in S$.

b). Suppose $(z,x) \in S$ and define the profile $\langle R_i \rangle$ by

$$R_i = \begin{cases} R_2 & i \in V \\ R_1 & i \in V^c. \end{cases}$$

Schematically we have

$\underline{V}$	$\underline{V^c}$
y	x
z	y
x	z.

From the Pareto principle y P z and since V is coercive for z over x, z P x. Thus y P x, so V is coercive for y over x, i.e., $(y,x) \in S$.

q.e.d.

The next step is to prove that if S is any set of pairs which is closed under coerciveness implications, then S is either empty or contains all pairs. It then follows from Lemma 1 that if a coalition is coercive for x over y, then it is coercive for any pair and hence coercive. Before proceeding we introduce some more notation.

Let $U^x(y) = \{z : |x - z| < |x - y|\}$, $L^x(y) = \{z : |x - z| > |x - y|\}$ and $H(x,y) = \{z : |x - z| < |y - z|\}$. Then $U^x(y)$ is just the inside of the sphere centered at x passing through y, $L^x(y)$ is everything outside the sphere, and H(x,y) is the half-space of everything closer to x than to y. See Figure 1.

If $x\ P^z\ y$, then $z \in H(x,y)$; if $z \in U^x\ (y)$, then $z\ P^x\ y$; and if $z \in L^x\ (y)$, then $y\ P^x\ z$.

[Figure 1 about here]

<u>Lemma 2</u>. Let S satisfy condition C., i.e., let S be closed under coerciveness implications. If $(a,b) \in S$ and $c \in L^a(b)$, then $(a,c) \in S$.

<u>Proof</u>. First consider the case where $c \in C_1 = [U^b(a) \cap L^a(b)]$. See Figure 2. Then $a\ P^a\ b\ P^a\ c$ and $b\ P^b\ c\ P^b\ a$. By condition C.a. (with $x = a$, $y = b$, $z = c$, $R_1 = R^a$, $R_2 = R^b$) we conclude $(a,c) \in S$. Thus starting with one pair $(a,b) \in S$ we now have a multitude. The next step is to expand the set even further. This part of the argument uses the fact that the space of alternatives is at least two-dimensional.

Choose a point $c_1 \in C_1$ as in Figure 2, that is, c_1 should nearly maximize the distance from b and minimize the distance from a. (Since C_1 is open, we cannot actually maximize or minimize these distances.) Repeating the above argument with c_1 in place of b yields $(a,c) \in S$ for all $c \in C_2 = [L^a(c_1) \cap U^{c_1}(a)]$.

Continue in this fashion to construct regions $C_k = [U^{c_{k-1}}(a) \cap L^a(c_{k-1})]$, which encompass the point a. We can also choose points like c_t, which is colinear with a and c_k for some k, to find a region C_{t+1} which includes points far away from a and such that $c \in C_{t+1}$ implies that $(a,c) \in S$.

Thus for any $x \in L^a(b)$ we can find a finite sequence of regions $C_1,\ldots,C_k$, constructed iteratively as above, such that $x \in C_k$, and hence $(a,x) \in S$.

q.e.d.

[Figure 2 about here]

<u>Lemma 3</u>. Let S satisfy condition C. If $(a,b) \in S$ and $c \in [L^a(b) \cap H(b,a)]$, then $(b,c) \in S$.

<u>Proof</u>. Put $w = (1/2)a + (1/2)b + \delta(b-a)$, where $\delta > 0$ is chosen small enough so that $b\ P^w\ a\ P^w\ c$. See Figure 3. We also have that $c\ P^c\ b\ P^c\ a$, as $c \in H(b,a)$. Then from condition C.b. (setting $x = c$, $y = b$, $z = a$, $R_1 = R^c$, $R_2 = R^w$) we have that $(b,c) \in S$.

q.e.d.

[Figure 3 about here]

<u>Lemma 4</u>. Let S satisfy condition C. If $(a,b) \in S$, then $(b,a) \in S$.

<u>Proof</u>. Let $(a,b) \in S$ and choose $c \in [L^a(b) \cap H(b,a) \cap U^b(a)]$. See Figure 4. Then by Lemma 3, $(b,c) \in S$. Since $c \in U^b(a)$ we have $a \in L^b(c)$, so by Lemma 2, $(b,a) \in S$.

q.e.d.

[Figure 4 about here]

<u>Lemma 5</u>. Let S satisfy condition C. Then S is either empty or includes all ordered pairs of distinct points.

<u>Proof</u>. Suppose S is nonempty and let $(a,b) \in S$. By Lemma 4, $(b,a) \in S$. Set $b' = a + \alpha(a - b)$, $\alpha > 1$. See Figure 5. Lemma 2 implies $(a,b') \in S$ so by Lemma 4 $(b',a) \in S$. But by construction $[L^{b'}(a) \cap H(a,b')] \cup [L^b(a) \cap H(a,b)] = \mathbb{R}^m \setminus \{a\}$, so by Lemma 3, for any $x \neq a$, $(a,x) \in S$ and so by Lemma 4, $(x,a) \in S$.

q.e.d.

[Figure 5 about here]

So far we have shown that if a coalition V is coercive for a over b for some (a,b), then it is coercive. Next we show that a coercive coalition is decisive.

<u>Lemma 6</u>. If V is coercive, then V is decisive.

<u>Proof</u>. Let $\langle R_i \rangle \in D^N$ be a profile with $x\ P_i\ y$ for all $i \in V$. Put $w_0 = (1/2)x + (1/2)y$, $w_1 = (5/8)x + (3/8)y$, $w_2 = (3/8)x + (5/8)y$, and $z = x + ½(x - y)$. See Figure 6. The ordering of x,y,z for the preferences R^x, R^{w_i}, $i = 0,1,2$ are then:

R^x	R^{w_0}	R^{w_1}	R^{w_2}
x	x y	x	y
z		y	x
y	z	z	z

[Figure 6 about here]

Define the profile $\langle R_i' \rangle$ by

$$R_i' = \begin{cases} R^x & i \in V \\ R^{w_1} & i \in V \text{ and } x \, P_i \, y \\ R^{w_2} & i \in V \text{ and } y \, P_i \, x \\ R^{w_0} & \text{otherwise.} \end{cases}$$

Then $x \, R_i' \, y \Longleftrightarrow x \, R_i \, y$ and so $x \, R \, y \Longleftrightarrow y \, R \, x$. By the Pareto principle $x \, P' \, z$. By construction $z \, P_i' \, y$ for $i \in V$ and $y \, P_i' \, z$ for $i \in V^c$. Since V is coercive, $z \, P' \, y$. By transitivity, $x \, P' \, y$ and hence $x \, P \, y$. Thus V is decisive.

q.e.d.

It follows from the Pareto principle that the coalition N is decisive. The following lemma shows the existence of other decisive coalitions.

<u>Lemma 7</u>. If V is not coercive, then V^c is coercive.

<u>Proof</u>. Suppose V is not coercive and let $\langle R_i \rangle$ be a profile with $x \, P_i \, y$ for all $i \in V$ and $y \, P_i \, x$ for all $i \in V^c$ and $y \, R \, x$. Put $z = x + (1/2)(x - y)$. By the condition of Independence of Irrelevant Alternatives we can assume without loss of generality that

$$R_i = \begin{cases} R^x & i \in V \\ R^y & i \in V^c. \end{cases}$$

Then $x \, P_i \, z$ for all $i \in N$ so $x \, P \, z$ by the Pareto principle. Since $y \, R \, x$, we have by transitivity that $y \, P \, z$. By construction $z \, P_i \, y$ for $i \in V$ and $y \, P_i \, z$ for $i \in V^c$. Thus V^c is coercive.

q.e.d.

The last piece of information we need to know about the collection of decisive coalitions is that the intersection of decisive coalitions is decisive. This argument relies heavily on the fact that the space of alternatives is multi-dimensional.

<u>Lemma 8</u>. If V_1 and V_2 are decisive, then $V_1 \cap V_2$ is decisive.

<u>Proof</u>. Choose alternatives a, b, c to be three vertices of an equilateral triangle. Put

$$x = (1/2 + \varepsilon)a + (1/2 - \varepsilon)b$$
$$y = (1/2 + \varepsilon)b + (1/2 - \varepsilon)c$$
$$z = (1/2 + \varepsilon)c + (1/2 - \varepsilon)a$$

where $\varepsilon > 0$ is small enough so that the following orderings hold:

$\underline{R^x}$	$\underline{R^y}$	$\underline{R^z}$
a	b	c
b	c	a
c	a	b

[Figure 7 about here]

See Figure 7. This is the classic paradox of voting situation. Define the profile $\langle R_i \rangle$ by

$$R_i = \begin{cases} R^y & i \in V_1 \setminus V_2 \\ R^x & i \in V_1 \cap V_2 \\ R^z & i \in V_2 \setminus V_1 \\ R^c & \text{otherwise.} \end{cases}$$

Schematically we have

$V_1 \setminus V_2$	$V_1 \cap V_2$	$V_2 \setminus V_1$	Others
b	a	c	c
c	b	a	a
a	c	b	

Since V_1 is decisive b P c, and since V_2 is decisive a P b. Thus by transivity a P c. By construction a P_i c for i $\in V \cap V_2$ and c P_i a for i $\in (V_1 \cap V_2)^c$. Thus $V_1 \cap V_2$ is coercive and hence decisive.

q.e.d.

<u>Proof of the Theorem</u>. Since N is finite, enumerate the voters from 1 to n. Assume that voters 1 to n-1 are not dictators. That is {j} is not a decisive coalition for j = 1,...,n-1. Then by Lemma 7, $\{j\}^c$ is decisive. It follows from Lemma 8 that $\{n\} = \bigcap_{j=1}^{n-1} \{j\}^c$ is decisive so voter n is a dictator.

Q.E.D.

<u>Remarks</u>. It is clear from the proof that any domain of preferences larger than type one preferences admits no nondictatorial Arrow-type social welfare function. Also, we have shown that the class of

decisive coalitions is an ultrafilter and so for infinite sets of voters there are nondictatorial Arrow-type social welfare functions (Fishburn [4], Hansson [5], Kirman and Sondermann [8]).

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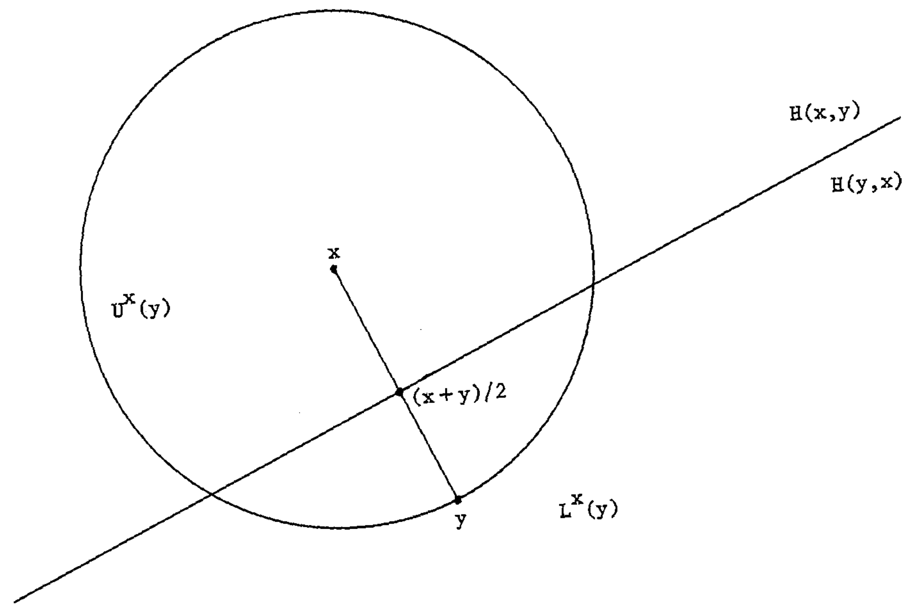


Figure 1

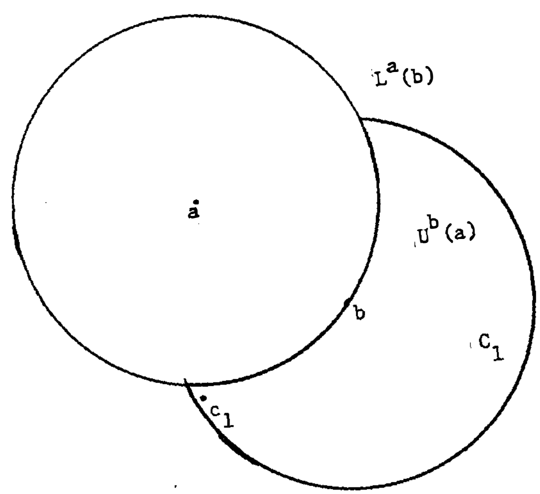
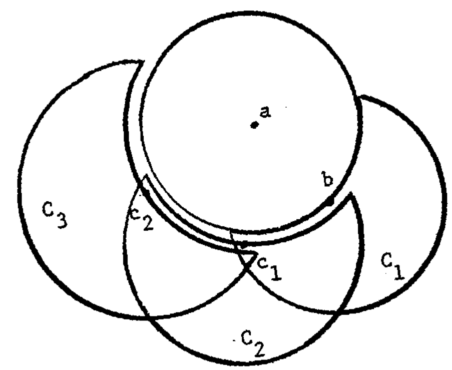


Figure 2a



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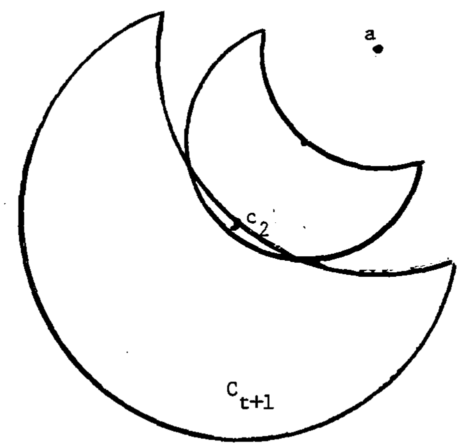

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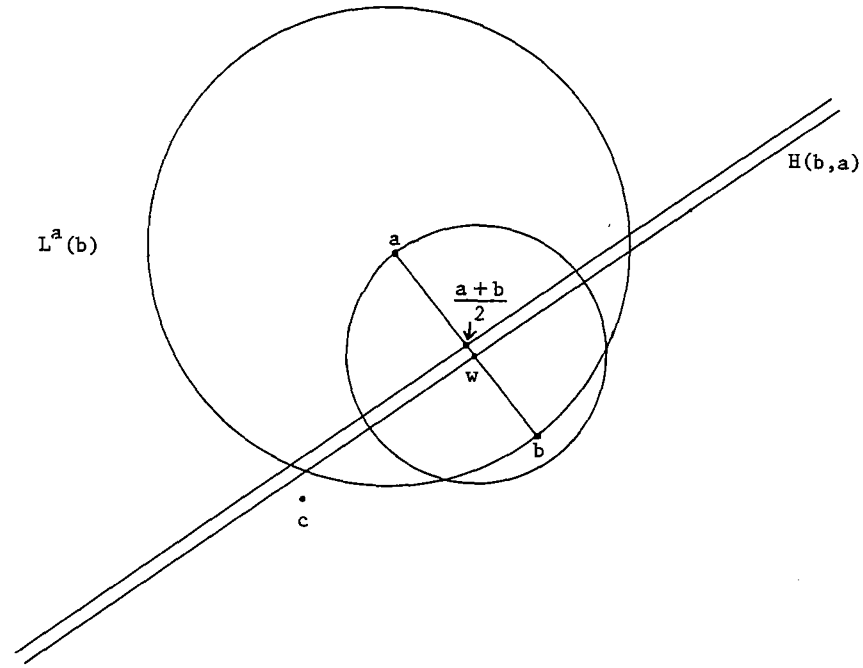

Figure 3

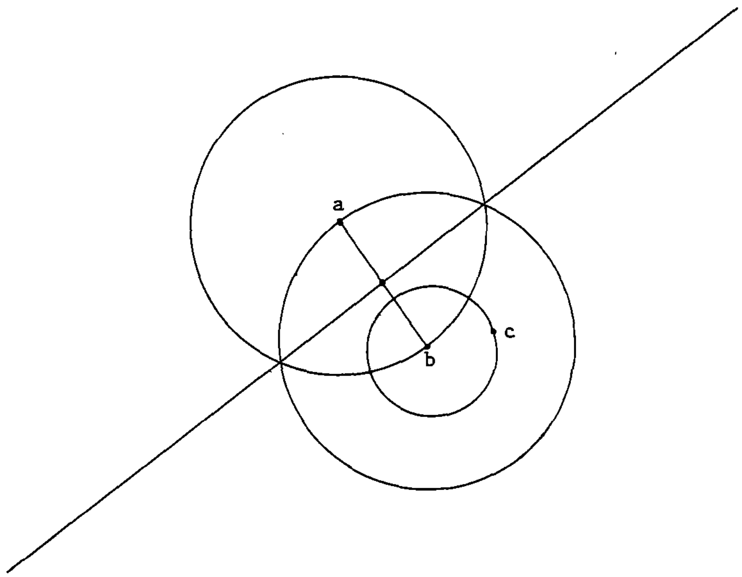

Figure 4

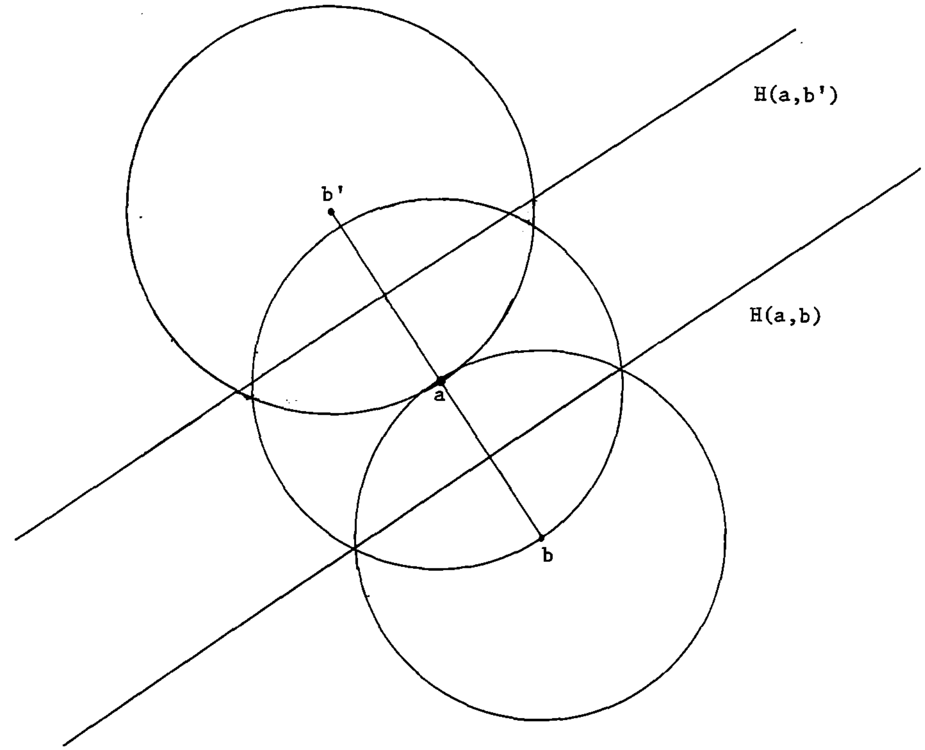

Figure 5

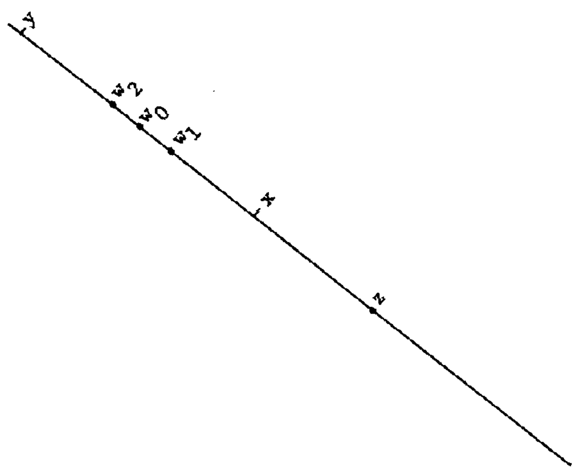

Figure 6

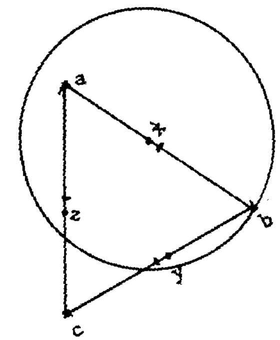

Figure 7